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STABILITY THEOREMS FOR THE BAROTROPIC VORTICITY EQUATION

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ABSTRACT

The conditions for uniqueness of solutions to the barotropic vorticity equation within a limited region are discussed, in particular for cases with flow through the boundary, when no physical boundary conditions exist. Two different sets of boundary conditions are given, for which the solution will remain uniquely defined as long as certain of its derivatives are bounded. A small perturbation on the initial solution will then also remain small, and the problem is thus properly posed. It is furthermore shown that similar conclusions may be drawn for the finite-difference vorticity equation of the "leapfrog" type, based on the symmetric-conservative Jacobian suggested by Arakawa and the normal five- or nine-point Laplacian, if in addition two stability conditions are satisfied, one of them being essentially the condition suggested by Charney, Fjörtoft, and von Neumann.

1. INTRODUCTION

The motions of large-scale disturbances in a rotating fluid may often be derived from the barotropic vorticity equation for a barotropic fluid

$$\frac{\partial}{\partial t}\nabla^2\psi=-J(\psi,\nabla^2\psi+f)+F+D(\psi) \qquad (1)$$

where ∇^2 and J are two-dimensional Laplace and Jacobi operators, F is a prescribed forcing function, $D(\psi)$ a dissipation term, usually of the type $-\kappa\nabla^2\psi$ (bottom-friction case) or $\nu\nabla^4\psi$ (diffusion case), ψ is the stream function for the flow, and f the coriolis parameter.

If the region of interest is a bounded basin, which may be plane or part of the surface of a sphere, the non-inflow boundary condition $\partial\psi/\partial s=0$ may be achieved by prescribing $\psi=0$ at the boundary. If the diffusion type of dissipation is used, we also have the nonslip condition $\partial\psi/\partial n=0$ at the boundary. In other cases, the solution is however assumed to exist at all points on the surface of a sphere, although we are only interested in its behavior within a restricted region. No physical arguments can then be used to derive a necessary set of boundary conditions. In their classical paper, Charney, Fjörtoft, and von Neumann (1950) concluded that ψ must be known at the whole boundary and $\nabla^2\psi$ at the inflow part of the boundary (where $\partial\psi/\partial s>0$), only on the basis of heuristic arguments.

In section 2 of this paper it will, however, be shown that a solution satisfying the Charney-Fjörtoft-von Neumann boundary conditions is actually uniquely determined, but that this is also the case if $\partial\psi/\partial n$ instead of $\nabla^2\psi$ is known at inflow points. The solution is then a continuous function of the initial data and the problem consequently properly posed. The proof is based on a stability function, giving an upper bound for the growth of perturbations on the correct solution to the equation. (For the basic ideas of this "energy method," see Richtmyer and Morton, 1967.) Only the plane case will be shown.

2. A STABILITY FUNCTION FOR THE BAROTROPIC VORTICITY EQUATION

Let $\psi=\psi^0(x,y,t)$ satisfy (1), certain yet unspecified boundary conditions on the boundary curve C of the plane region R, and the initial condition $\psi(x,y,0)=\psi^0(x,y,0)$. With slightly perturbed initial and boundary values, the solution would be $\psi^0(x,y,t)+\psi'(x,y,t)$, the disturbance ψ' satisfying the equation

$$\frac{\partial}{\partial t}\nabla^2\psi'=-J(\psi',\eta^0)-J(\psi^0+\psi',\nabla^2\psi')+D(\psi') \qquad (2)$$

where $\eta^0=\nabla^2\psi^0+f$.

A suitable measure of the intensity of the disturbance is its root-mean-square vorticity or root-mean-square velocity. Using the inner product symbols

$$[\alpha,\beta]=\int_R \alpha\beta\, dS,\ [\nabla\alpha,\nabla\beta]=\int_R \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x}+\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}\,dS,$$
$$||\alpha||=[\alpha,\alpha]^{1/2},\text{ and } ||\nabla\alpha||=[\nabla\alpha,\nabla\alpha]^{1/2},$$

the symmetry property of the Jacobian

$$J(\alpha,\beta)=-J(\beta,\alpha), \qquad (3)$$

and the integral relations

$$[\alpha,\nabla^2\beta]+[\nabla\alpha,\nabla\beta]=\oint_C \alpha\frac{\partial\beta}{\partial n}\,ds \qquad (4)$$

and

$$[\gamma,J(\alpha,\beta)]+[\beta,J(\alpha,\gamma)]=-\oint_C \beta\gamma\frac{\partial\alpha}{\partial s}\,ds, \qquad (5)$$

we get the following equations for the growth rates of these norms. For the RMS vorticity,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^2\psi'\|^2&=-[\nabla^2\psi', J(\psi',\eta^0)+J(\psi^0+\psi',\nabla^2\psi')]+[\nabla^2\psi', D(\psi')]\\&=\oint_C\frac{1}{2}(\nabla^2\psi')^2\frac{\partial}{\partial s}(\psi^0+\psi')ds+[\nabla^2\psi', J(\eta^0,\psi')]+[\nabla^2\psi', D(\psi')],\end{aligned}\quad(6)$$

and for the RMS velocity

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\psi'\|^2&=\oint_C\psi'\frac{\partial^2\psi'}{\partial n\partial t}ds-[\psi',\frac{\partial}{\partial t}\nabla^2\psi']\\&=\oint_C\psi'\frac{\partial^2\psi'}{\partial n\partial t}ds+[\psi', J(\psi',\eta^0)+J(\psi^0+\psi',\nabla^2\psi')]-[\psi', D(\psi')]\\&=\oint_C\psi'\frac{\partial^2\psi'}{\partial n\partial t}+\frac{1}{2}\psi'^2\frac{\partial\eta^0}{\partial s}-\psi'\nabla^2\psi'\frac{\partial}{\partial s}(\psi^0+\psi')ds\\&\quad-[\nabla^2\psi', J(\psi^0,\psi')]-[\psi', D(\psi')].\end{aligned}\quad(7)$$

To simplify the treatment, consider only the bottom-friction type of dissipation $D(\psi')=-\kappa\nabla^2\psi'$. Then, $[\nabla^2\psi', D(\psi')]=-\kappa\|\nabla^2\psi'\|^2$ and

$$-[\psi', D(\psi')]=\oint_C\kappa\psi'\frac{\partial\psi'}{\partial n}ds-\kappa\|\nabla\psi'\|^2.$$

We must now derive upper bounds for the inner products $[\nabla^2\psi', J(\eta^0,\psi')]$ and $[\nabla^2\psi', J(\psi^0,\psi')]$. This can be done in three ways.

1) From the definition of the Jacobian and the well-known inequality $2xy\le\frac{1}{a}x^2+ay^2$ (a is an arbitrary positive number) follows that an inner product of the type $[\gamma, J(\alpha,\beta)]$ satisfies the relation

$$[\gamma, J(\alpha,\beta)]\le[|\gamma|, |J(\alpha,\beta)|]\le[|\gamma|, |\nabla\alpha||\nabla\beta|]\le\max_R|\nabla\alpha|\,[|\gamma|,|\nabla\beta|]\le\max_R|\nabla\alpha|\left(\frac{1}{2a}\|\gamma\|^2+\frac{a}{2}\|\nabla\beta\|^2\right)\quad(8)$$

where

$$|\nabla\alpha|=((\partial\alpha/\partial x)^2+(\partial\alpha/\partial y)^2)^{1/2}.$$

2) By partial integration, an inner product of the type $[\nabla^2\beta, J(\alpha,\beta)]$ can be transformed to yield

$$\begin{aligned}[\nabla^2\beta, J(\alpha,\beta)]=&-\oint_C\beta J\left(\alpha,\frac{\partial\beta}{\partial n}\right)+\left(\frac{1}{2}(\nabla\beta)^2+\beta\nabla^2\beta\right)\frac{\partial\alpha}{\partial s}\\&+\beta\nabla\beta\cdot\nabla\frac{\partial\alpha}{\partial s}ds+\int_R\left(\left(\frac{\partial\beta}{\partial x}\right)^2-\left(\frac{\partial\beta}{\partial y}\right)^2\right)\frac{\partial^2\alpha}{\partial x\,\partial y}\\&+\frac{\partial\beta}{\partial x}\frac{\partial\beta}{\partial y}\left(\frac{\partial^2\alpha}{\partial y^2}-\frac{\partial^2\alpha}{\partial x^2}\right)dS,\end{aligned}\quad(9)$$

and the latter integral is less than $\frac{1}{2}\mathcal{D}(\alpha)\|\nabla\beta\|^2$ where $\mathcal{D}(\alpha)$ is the total deformation for the field α.

3) The third type of upper bound is a modification of type 1). It uses the fact that for some λ_{max},

$$\|\nabla\beta\|^2\le\lambda_{max}^2\|\nabla^2\beta\|^2\quad(10)$$

if $\beta=0$ at the boundary of the closed region R, so that

$$[\nabla^2\beta, J(\alpha,\beta)]\le\lambda_{max}\max_R|\nabla\alpha|\,\|\nabla^2\beta\|^2.\quad(11)$$

With the first type of estimate, we obtain from (6), (7), and (8)

$$\begin{cases}\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^2\psi'\|^2+\kappa\|\nabla^2\psi'\|^2\le I_1+\frac{1}{2}\max_R|\nabla\eta^0|\times\left(\frac{1}{a}\|\nabla^2\psi'\|^2+a\|\nabla\psi'\|^2\right) & (12)\\ \frac{1}{2}\frac{\partial}{\partial t}\|\nabla\psi'\|^2+\kappa\|\nabla\psi'\|\nabla^2\le I_2+\frac{1}{2}\max_R|\nabla\psi^0|\times\left(\frac{1}{a}\|\nabla^2\psi'\|^2+a\|\nabla\psi'\|^2\right) & (13)\end{cases}$$

where

$$I_1=\oint_C\frac{1}{2}(\nabla^2\psi')^2\frac{\partial}{\partial s}(\psi^0+\psi')ds,\ I_2=\oint_C\psi'\frac{\partial^2\psi'}{\partial n\partial t}+\frac{1}{2}\psi'^2\frac{\partial\eta^0}{\partial s}-\psi'\nabla^2\psi'\frac{\partial}{\partial s}(\psi^0+\psi')+\kappa\psi'\frac{\partial\psi'}{\partial n}ds,$$

or if

$$K_1=\max_{0\le t\le T}(\max_R|\nabla\eta^0|)^{1/2},\ K_2=\max_{0\le t\le T}(\max_R|\nabla\psi^0|)^{1/2},$$

$$\frac{1}{2}\frac{\partial}{\partial t}(K_2^2\|\nabla^2\psi'\|^2+K_1^2\|\nabla\psi'\|^2)\le K_2^2I_1+K_1^2I_2+(K_1K_2-\kappa)(K_2^2\|\nabla^2\psi'\|^2+K_1^2\|\nabla\psi'\|^2).\quad(14)$$

The second type of estimate (9) gives with (7)

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\psi'\|^2\le I_3+(K_3-\kappa)\|\nabla\psi'\|^2\quad(15)$$

with

$$I_3=I_2+\oint_C\psi'J\left(\psi^0,\frac{\partial\psi'}{\partial n}\right)+\left(\frac{1}{2}(\nabla\psi')^2+\psi'\nabla^2\psi'\right)\frac{\partial\psi^0}{\partial s}+\psi'\nabla\psi'\cdot\nabla\frac{\partial\psi'}{\partial s}ds$$

and

$$K_3=\frac{1}{2}\max_{0\le t\le T}\left(\max_R\mathcal{D}(\psi^0)\right),$$

and the third type (11) combines with (6) into

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^2\psi'\|^2 \le I_1+(K_4-\kappa)\|\nabla^2\psi'\|^2 \quad (16)$$

where

$$K_4=\lambda_{max}\max_{0\le t\le T}\left(\max_R|\nabla\eta^0|\right).$$

Upper bounds for the norms are now easy to give if the boundary conditions make the integrals $K_2^2I_1+K_1^2I_2$, I_3, or I_1 nonpositive. One such set of boundary conditions is $\psi'=0$; $\nabla^2\psi'=0$ when $\frac{\partial\psi^0}{\partial s}\ge 0$, making $I_2=0$, $I_1\le 0$. Equation (14) then gives

$$K_2^2\|\nabla^2\psi'\|^2+K_1^2\|\nabla\psi'\|^2 \le (K_2^2\|\nabla^2\psi'\|^2+K_1^2\|\nabla\psi'\|^2)_{t=0}\exp(2(K_1K_2-\kappa)t) \quad (17)$$

or simply, using (10),

$$\|\nabla^2\psi'\|^2 \le \left(1+\frac{K_1^2}{K_2^2}\lambda_{max}^2\right)\|\nabla^2\psi'\|^2_{t=0}\cdot\exp(2(K_1K_2-\kappa)t). \quad (18)$$

Alternatively, (16) gives

$$\|\nabla^2\psi'\|^2\le\|\nabla^2\psi'\|^2_{t=0}\cdot\exp(2(K_4-\kappa)t). \quad (19)$$

This is not the only useful set of boundary conditions. If $\psi'=0$ at the whole boundary and $\frac{\partial\psi'}{\partial n}=0$ when $\frac{\partial\psi^0}{\partial s}\ge 0$, $I_2=0$ and $I_3\le 0$ so that

$$\|\nabla\psi'\|^2\le\|\nabla\psi'\|^2_{t=0}\cdot\exp(2(K_3-\kappa)t). \quad (20)$$

From these inequalities two important conclusions may be drawn:

a) If the initial perturbation $\psi'(x,y,0)=0$ and the first type of boundary condition is used, we obtain $\|\nabla^2\psi'\|^2=0$ for all $t>0$, as long as K_1 K_2 or K_4 is bounded. Since $\psi'=0$ at the boundary, this implies that the disturbance itself remains identically equal to zero, and it is thus impossible to obtain more than one solution to the equation from given initial data. With the second type of boundary condition, we obtain the same result concerning the uniqueness of the solution if K_3 is bounded, since $\psi'(x,y,0)=0$ then implies $\|\nabla\psi'\|^2=0$ and thus $\psi'=0$ for all $t>0$.

b) Any one of these inequalities gives an upper bound for the growth rate of a certain norm of ψ' which is only a function of the undisturbed solution. This guarantees that the problem is properly posed and that it is meaningful to search for a solution by a finite-difference technique.

3. A STABILITY THEOREM FOR THE FINITE-DIFFERENCE BAROTROPIC VORTICITY EQUATION

The numerical method suggested by Charney et al. (1950) for the integration of the plane, frictionless version of (1) was the central-difference "leapfrog" scheme

$$\nabla^2_{(5)}\Psi^{k+1}_{i,j}-\nabla^2_{(5)}\Psi^{k-1}_{i,j}=-2\Delta t J_{(5)}(\Psi^k,\nabla^2_{(5)}\Psi^k+f)_{i,j} \quad (21)$$

with the finite-difference Laplace and Jacobi operators chosen as

$$\nabla^2_{(5)}\Psi_{i,j}\equiv\frac{1}{\Delta s^2}(\Psi_{i+1,j}+\Psi_{i-1,j}+\Psi_{i,j+1}+\Psi_{i,j-1}-4\Psi_{i,j}) \quad (22)$$

and

$$J_{(5)}(\alpha,\beta)_{i,j}\equiv\frac{1}{4\Delta s^2}((\alpha_{i+1,j}-\alpha_{i-1,j})(\beta_{i,j+1}-\beta_{i,j-1}) -(\alpha_{i,j+1}-\alpha_{i,j-1})(\beta_{i+1,j}-\beta_{i-1,j}))=J^{++}(\alpha,\beta)_{i,j}. \quad (23)$$

From a linearized disturbance equation with locally constant coefficients, they concluded that the solution should be stable for all Δt less than a certain $\Delta t_{min}(\Delta s)$. As shown by Phillips (1959), this conclusion is not valid for finite disturbances. Due to the limited accuracy of the initial data and other errors, an unlimited error growth may thus set in after a finite integration time. By changing the finite-difference Jacobian to the symmetric conservative form

$$\begin{aligned}J^1_{(9)}(\alpha,\beta)_{i,j}\equiv&\tfrac{1}{3}\{D_{0,x}(\alpha D_{0,y}\beta)-D_{0,y}(\alpha D_{0,x}\beta)+D_{0,y}(\beta D_{0,x}\alpha)\\&-D_{0,x}(\beta D_{0,y}\alpha)+D_{0,x}\alpha\cdot D_{0,y}\beta-D_{0,y}\alpha\cdot D_{0,x}\beta\}_{i,j}\\&=\tfrac{1}{3}\{J^{+\times}(\alpha,\beta)_{i,j}+J^{\times+}(\alpha,\beta)_{i,j}+J^{++}(\alpha,\beta)_{i,j}\}\end{aligned} \quad (24)$$

where $D_{0,x}$ and $D_{0,y}$ are central-difference operators defined by

$$D_{0,x}\alpha_{i,j}\equiv\frac{1}{2\Delta s}(\alpha_{i+1,j}-\alpha_{i-1,j}),\ D_{0,y}\alpha_{i,j}\equiv\frac{1}{2\Delta s}(\alpha_{i,j+1}-\alpha_{i,j-1}). \quad (25)$$

Arakawa (1966) constructed a scheme where the errors could not grow without limit, although it was not shown whether or when they would remain small compared to the exact solution. The question of stability conditions was not discussed.

A detailed investigation of the computational stability of the finite-difference barotropic vorticity equation

$$\frac{1}{2\Delta t}(\nabla^2\Psi^{k+1}_{i,j}-\nabla^2\Psi^{k-1}_{i,j})=-J(\Psi^k,\nabla^2\Psi^k+f)_{i,j}+F^k_{i,j} +D\left(\frac{1}{2}(\Psi^{k+1}_{i,j}+\Psi^{k-1}_{i,j})\right) \quad (26)$$

will here be presented for the case when R is a rectangular region, assuming the Jacobian to be of the symmetric-conservative form (24) introduced by Arakawa. A similar result may be derived for a region covered by a rhombic (hexagonal)-rectagonal grid with the Jacobian suggested by Sadourny et al. (1968). The idea of the proof is similar to the one leading to the estimate (19).

Consider for simplicity only the case $D(\psi)=-\kappa\nabla^2\psi$ and let R be a rectangle with sides $L\Delta s$ and $M\Delta s$. Ψ is thus defined on the grid (x_i, y_j, t_k), $0\leq i\leq L$, $0\leq j\leq M$, $0\leq k\leq N=T/\Delta t$ where the sets of points with $i=0$ or L and with $j=0$ or M are the boundary points. At the boundary, both Ψ and $\nabla^2\Psi$ will be assumed to be known, although (19) did only require a knowledge of $\nabla^2\psi$ at the inflow part of the boundary. This assumption is made partly because of its common use, partly because of the resulting simplifications. Some numerical boundary condition must be used at the outflow part of the boundary, but the choice of this additional condition should not influence the stability and accuracy very much. (See note added in proof on page 345.)

A suitable inner product for scalars is simply

$$[\alpha,\beta]\equiv\sum_{i=1}^{L-1}\sum_{j=1}^{M-1}\Delta s^2\alpha_{i,j}\beta_{i,j}. \tag{27}$$

The definition of an inner product corresponding to $[\nabla\alpha, \nabla\beta]$ depends on the type of Laplace operator being used. Since we are not considering schemes using special formulae for points near the boundary, all operators must be of the simplest nine-point type. The Laplacian could then be either the five-point operator $\nabla^2_{(5)}$ given by (22), the nine-point operator $\nabla^2_{(9)}$ defined by

$$\nabla^2_{(9)}\Psi_{i,j}=\frac{1}{6\Delta s^2}(\Psi_{i+1,j+1}+\Psi_{i+1,j-1}+\Psi_{i-1,j+1}+\Psi_{i-1,j-1}+4(\Psi_{i+1,j}+\Psi_{i-1,j}+\Psi_{i,j+1}+\Psi_{i,j-1})-20\Psi_{i,j}), \tag{28}$$

or a combination of these. The inner product $[\nabla\alpha, \nabla\beta]$ should satisfy the relation

$$[\nabla\alpha,\nabla\beta]=-[\alpha,\nabla^2\beta] \tag{29}$$

if $\alpha=0$ at the boundary. By partial summation, it is easily shown that this is true for

$$[\nabla_{(5)}\alpha,\nabla_{(5)}\beta]\equiv\sum_{i=0}^{L-1}\sum_{j=0}^{M-1}\Delta s^2(D_{+x}\alpha_{i,j}\cdot D_{+x}\beta_{i,j}+D_{+y}\alpha_{i,j}\cdot D_{+y}\beta_{i,j}) \tag{30}$$

and for

$$[\nabla_{(9)}\alpha, \nabla_{(9)}\beta]\equiv\frac{2}{3}[\nabla_{(5)}\alpha, \nabla_{(5)}\beta]+\frac{1}{3}\sum_{i=0}^{L-1}\sum_{j=0}^{M-1}\Delta s^2(D_{+\xi}\alpha_{i,j}\cdot D_{+\xi}\beta_{i,j}+D_{+\eta}\alpha_{i+1,j}\cdot D_{+\eta}\beta_{i+1,j}) \tag{31}$$

where D_+ and D_- denote the forward and backward difference operators in the x, y, and the two diagonal (ξ,η) directions, e.g.,

$$D_{+x}\psi_{i,j}\equiv\frac{1}{\Delta s}(\psi_{i+1,j}-\psi_{i,j});\ D_{+\xi}\psi_{i,j}\equiv\frac{1}{\sqrt{2}\Delta s}(\psi_{i+1,j+1}-\psi_{i,j});\ D_{+\eta}\psi_{i,j}\equiv\frac{1}{\sqrt{2}\Delta s}(\psi_{i-1,j+1}-\psi_{i,j}).$$

We want the Jacobian to satisfy relations similar to (3) and (5); the only possible form is thenAraka wa's $J^1_{(9)}(\alpha,\beta)$. The relation

$$J^1_{(9)}(\alpha,\beta)=-J^1_{(9)}(\beta,\alpha) \tag{32}$$

follows by inspection, and summation by parts gives

$$[\gamma, J^1_{(9)}(\alpha,\beta)]=-[\beta, J^1_{(9)}(\alpha,\gamma)] \tag{33}$$

provided $\beta=\gamma=0$ at the boundary.

The perturbation equation is now

$$(1+\kappa\Delta t)\nabla^2\Psi'^{k+1}-(1-\kappa\Delta t)\nabla^{2\prime}\Psi^{k-1}=-2\Delta t(J(\Psi'^k,\eta^{0k})+J(\Psi^{0k}+\Psi'^k,\nabla^2\Psi'^k)) \tag{34}$$

where $\eta^0=\nabla^2\Psi^0+f$ and the indices i, j have been dropped. Introduce now the magnified disturbance function

$$\varphi'^k=\left(\frac{1+\kappa\Delta t}{1-\kappa\Delta t}\right)^{k/2}\Psi'^k,$$

satisfying the simpler equation

$$\nabla^2\varphi'^{k+1}-\nabla^2\varphi'^{k-1}=-2\Delta' t(J(\varphi'^k,\eta^{0k})+J(\Psi^{0k}+\Psi'^k,\nabla^2\varphi'^k)) \tag{35}$$

where $\Delta' t\equiv(1-\kappa^2\Delta t^2)^{-1/2}\Delta t$.

Taking the inner product with $\nabla^2\varphi'^{k+1}+\nabla^2\varphi'^{k-1}$,

$$||\nabla^2\varphi'^{k+1}||^2-||\nabla^2\varphi'^{k-1}||^2=-2\Delta' t[\nabla^2\varphi'^{k+1}+\nabla^2\varphi'^{k-1}, J(\varphi'^k,\eta^{0k})+J({}^0\Psi^k+\Psi'^k,\nabla^2\varphi'^k)] \tag{36}$$

or

$$L^{k+1}-L^k=-2\Delta' t(M^k{}_1+M^k{}_2) \tag{37}$$

where

$$L^k=||\nabla^2\varphi'^k||^2+||\nabla^2\varphi'^{k-1}||^2+2\Delta' t[\nabla^2\varphi'^k, J(\varphi'^{k-1},\eta^{0k-1})+J(\Psi^{0k-1}+\Psi'^{k-1},\nabla^2\varphi'^{k-1})],$$

$$M^k{}_1=[\nabla^2\varphi'^k, J(\varphi'^{k-1},\eta^{0k-1})]+[\nabla^2\varphi'^{k-1}, J(\varphi'^k,\eta^{0k})],$$

and

$$M^k{}_2=[\nabla^2\varphi'^k, J(\Psi^{0k-1}+\Psi'^{k-1},\nabla^2\varphi'^{k-1})]+[\nabla^2\varphi'^{k-1}, J(\Psi^{0k}+\Psi'^k,\nabla^2\varphi'^k)].$$

Then,

$$L^k=L^1-2\Delta' t\sum_{\nu=1}^{k-1}(M^\nu{}_1+M^\nu{}_2)\leq L^1+2\Delta' t\sum_{\nu=1}^{k-1}(|M^\nu{}_1|+|M^\nu{}_2|). \tag{38}$$

As above, we need upper bounds for inner products of the type $[\gamma, J(\alpha,\beta)]$.

In the appendix is shown that

$$[\gamma, J^1_{(9)}(\alpha,\beta)]\leq\frac{1}{\sqrt{2}}G(\alpha)\left(\frac{1}{a}||\gamma||^2+a||\nabla\beta||^2\right) \tag{39}$$

and

$$[\gamma, J^{1}_{(9)}(\alpha, \beta)] \leq \frac{1}{\Delta s} G(\alpha) \left(\frac{1}{a}||\gamma||^2 + a||\beta||^2\right) \tag{40}$$

where

$G(\alpha) \equiv \max_{i,j}(max(|D_{+x}\alpha_{i,j}|, |D_{+y}\alpha_{i,j}|))$ if the Laplacian $\nabla^2_{(5)}$ is used and $G(\alpha) \equiv \max_{i,j}(max(|D_{+x}\alpha_{i,j}|, |D_{+y}\alpha_{i,j}|, |D_{+\xi}\alpha_{i,j}|, |D_{+\eta}\alpha_{i,j}|))$ if $\nabla^2_{(9)}$ is used. Also, $||\nabla\varphi'|| \leq \lambda_{max}||\nabla^2\varphi'||$ for some λ_{max}, so that we obtain the bounds

$$L^k \geq \left(1 - 2\frac{\Delta' t}{\Delta s} G(\Psi^{0^{k-1}} + \Psi'^{k-1}) - \sqrt{2}\Delta' t \lambda_{max} G(\eta^{0^{k-1}})\right) \times (||\nabla^2\varphi'^k||^2 + ||\nabla^2\varphi'^{k-1}||^2), \tag{41}$$

$$M_1^\nu \leq \frac{1}{\sqrt{2}} \lambda_{max}(G(\eta^{0^{\nu-1}}) + G(\eta^{0^\nu}))(||\nabla^2\varphi'^\nu||^2 + ||\nabla^2\varphi'^{\nu-1}||^2), \tag{42}$$

and

$$M_2^\nu \leq \frac{1}{\Delta s} G(\Psi^{0^\nu} + \Psi'^\nu - \Psi^{0^{\nu-1}} - \Psi'^{\nu-1})(||\nabla^2\varphi'^\nu||^2 + ||\nabla^2\varphi'^{\nu-1}||^2). \tag{43}$$

If we can now find a constant C_1 so that

$$1 - 2\frac{\Delta' t}{\Delta s} \max_{0 \leq \nu \leq N} \left(G(\Psi^{0^\nu} + \Psi'^\nu) + \frac{1}{\sqrt{2}} \Delta s \lambda_{max} G(\eta^{0^\nu})\right) \geq C_1 > 0 \tag{44}$$

and the solution changes so little from time step to time step that

$$\max_{0 \leq \nu \leq N} G\left(\frac{1}{\Delta t}(\Psi^{0^\nu} + \Psi'^\nu - \Psi^{0^{\nu-1}} - \Psi'^{\nu-1})\right) \leq C_2\sqrt{2}\frac{\Delta s}{\Delta t} \cdot \lambda_{max} \max_{0 \leq \nu \leq N} G(\eta^{0^\nu}) \tag{45}$$

where C_2 does not depend upon the choice of Δt and Δs, we have

$$||\nabla^2\varphi'^k||^2 + ||\nabla^2\varphi'^{k-1}||^2 \leq \frac{L^1}{C_1} + 2\Delta' t \frac{C_2+1}{C_1}\sqrt{2}\lambda_{max} \cdot \max_\nu G(\eta^{0^\nu}) \sum_{\nu=1}^{k-1} (||\nabla^2\varphi'^\nu||^2 + ||\nabla^2\varphi'^{\nu-1}||^2)$$
$$\leq \left(1 + 2\sqrt{2}\frac{C_2+1}{C_1}\Delta' t \lambda_{max} \cdot \max_\nu G(\eta^{0^\nu})\right)^{k-1} \frac{L^1}{C_1}. \tag{46}$$

From this inequality and the upper bound for L^1,

$$L^1 \leq (2 - C_1)(||\nabla^2\varphi'^1||^2 + ||\nabla^2\varphi'^0||^2), \tag{47}$$

we finally obtain, in terms of the original variable Ψ',

$$||\nabla^2\Psi'^k||^2 \leq \frac{2-C_1}{C_1}\left(\frac{1-\kappa\Delta t}{1+\kappa\Delta t}\right)^k \left(1 + 2\sqrt{2}\frac{C_2+1}{C_1}\Delta' t \lambda_{max} \cdot \max_\nu G(\eta^{0^\nu})\right)^{k-1} (||\nabla^2\Psi'^1||^2 + ||\nabla^2\Psi'^0||^2)$$
$$\leq \frac{2-C_1}{C_1} \exp\left(2\left(\frac{C_2+1}{C_1}\sqrt{2}\lambda_{max} \cdot \max_\nu G(\eta^{0^\nu}) - \kappa + 0(\Delta t)\right) k\Delta t\right)(||\nabla^2\Psi'^1||^2 + ||\nabla^2\Psi'^0||^2). \tag{48}$$

For all $k \leq N = T/\Delta t$, this gives a bound for the norm of $\nabla^2\Psi'^k$ in terms of the corresponding norms of the initial disturbances $\nabla^2\Psi'^1$ and $\nabla^2\Psi'^0$. This proves the stability of the leapfrog scheme with the symmetric conservative Jacobian as long as (44) and (45) are satisfied.

Of the stability conditions (44) and (45), the first one is essentially the same as Charney-Fjörtoft-von Neumann's heuristic stability condition. Since they are written in terms of the disturbed stream function and the undisturbed vorticity fields, they have an *a posteriori* character. If they are satisfied for small perturbations, stability prevails also for finite disturbances as long as

1) $\max_\nu G(\eta^{0^\nu}) \ll \lambda_{max}^{-1}\Delta s^{-1} \max_\nu G(\Psi^{0^\nu} + \Psi'^\nu)$

or

2) $\max_\nu G(\eta^{0^\nu}) \approx \max_\nu G(\eta^{0^\nu} + \eta'^\nu)$,

but since the exact solution is unknown, it is recommended to choose Δt safely below the upper bound set by the stability conditions determined from an approximate solution.

4. CONCLUSIONS

It has been shown that the problem of integrating the barotropic vorticity equation within a restricted region is actually properly posed if boundary conditions of the Charney-Fjörtoft-von Neumann type are used, but that is also true for a different set of conditions. The solution will be unique as long as certain derivatives of it remain bounded, and a small error in the initial data cannot grow with an unlimited speed. The same is true for the finite-difference form of the equation, based on the symmetric conservative Jacobian, at least if the region is rectangular and both Ψ and $\nabla^2\Psi$ are known at the boundary. Letting Δt and Δs approach zero in a way that keeps the stability conditions satisfied, it is also possible to show that the solution actually converges to the exact solution of the barotropic vorticity equation. These conclusions remain valid as long as certain derivatives of the solution remain bounded. It is, however, not possible to say in advance whether they will be true for all $t \geq 0$, which is an effect of the nonlinearity of the equation, but it is in most cases likely that the presence of a dissipation term should prevent the occurrence of unlimited values for these derivatives.

5. APPENDIX

DERIVATION OF UPPER BOUNDS

To derive upper bounds for $[\gamma, J^1_{(9)}(\alpha,\beta)]$ when $\nabla^2_{(5)}$ is used, we may write the symmetric conservative Jacobian as

$$J^1_{(9)}(\alpha, \beta)_{i,j} = \tfrac{1}{6}\{D_{+x}\alpha_{i,j}D_{0,y}\beta_{i+1,j} + D_{+x}\alpha_{i-1,j}D_{0,y}\beta_{i-1,j} - D_{+y}\alpha_{i,j}D_{0,x}\beta_{i,j+1} - D_{+y}\alpha_{i,j-1}D_{0,x}\beta_{i,j-1} + D_{+y}\beta_{i,j}D_{0,x}\alpha_{i,j+1} + D_{+y}\beta_{i,j-1}D_{0,x}\alpha_{i,j-1} - D_{+x}\beta_{i,j}D_{0,y}\alpha_{i+1,j} - D_{+x}\beta_{i-1,j}D_{0,y}\alpha_{i-1,j} + 2D_{0,x}\alpha_{i,j}D_{0,y}\beta_{i,j} - 2D_{0,y}\alpha_{i,j}D_{0,x}\beta_{i,j}\} \tag{49}$$

or alternatively as

$$\mathcal{J}^1_{(9)}(\alpha,\beta)_{i,j}=\tfrac{1}{6}\{D_{+x}\alpha_{i,j}D_{0,y}\beta_{i+1,j}+D_{+x}\alpha_{i-1,j}D_{0,y}\beta_{i-1,j}$$

$$-D_{+y}\alpha_{i,j}D_{0,x}\beta_{i,j+1}-D_{+y}\alpha_{i,j-1}D_{0,x}\beta_{i,j-1}+\frac{1}{\Delta s}(\beta_{i,j+1}D_{0,x}\alpha_{i,j+1}$$

$$-\beta_{i,j-1}D_{0,x}\alpha_{i,j-1}+\beta_{i+1,j}D_{0,y}\alpha_{i+1,j}-\beta_{i-1,j}D_{0,y}\alpha_{i-1,j})$$

$$+2D_{0,x}\alpha_{i,j}D_{0,y}\beta_{i,j}-2D_{0,y}\alpha_{i,j}D_{0,x}\beta_{i,j}\}. \quad (50)$$

From (49), we then get

$$[\gamma,\mathcal{J}^1_{(9)}(\alpha,\beta)]=\sum_{i=1}^{L-1}\sum_{j=1}^{M-1}\Delta s^2\gamma_{i,j}\mathcal{J}^1_{(9)}(\alpha,\beta)_{i,j}\leq\sum_{i=1}^{L-1}\sum_{j=1}^{M-1}\Delta s^2|\gamma_{i,j}|$$

$$|\mathcal{J}^1_{(9)}(\alpha,\beta)|\leq G_{(5)}(\alpha)\sum_{i=1}^{L-1}\sum_{j=1}^{M-1}\Delta s^2|\gamma_{i,j}|\tfrac{1}{6}\{|D_{0,y}\beta_{i+1,j}|+|D_{0,y}\beta_{i-1,j}|$$

$$+|D_{0,x}\beta_{i,j+1}|+|D_{0,x}\beta_{i,j-1}|+|D_{+y}\beta_{i,j}|+|D_{+y}\beta_{i,j-1}|+|D_{+x}\beta_{i,j}|$$

$$+|D_{+x}\beta_{i-1,j}|+2|D_{0,y}\beta_{i,j}|+2|D_{0,x}\beta_{i,j}|\} \quad (51)$$

with

$$G_{(5)}(\alpha)\equiv\max_{i,j}(\max(|D_{+x}\alpha_{i,j}|,|D_{+y}\alpha_{i,j}|)). \quad (52)$$

If $\beta=0$ at the boundary,

$$\sum\sum\Delta s^2|\gamma_{i,j}|\cdot|D_{0,y}\beta_{i+1,j}|\leq\frac{1}{\sqrt{2}}\sum\sum\Delta s^2\left(\frac{1}{2a}|\gamma_{i,j}|^2+a|D_{0,y}\beta_{i+1,j}|^2\right)$$

$$=\frac{1}{\sqrt{2}}\sum\sum\Delta s^2\left(\frac{1}{2a}|\gamma_{i,j}|^2+a|D_{0,y}\beta_{i,j}|^2\right)\text{ etc.}$$

so that

$$[\gamma,\mathcal{J}^1_{(9)}(\alpha,\beta)]\leq\frac{1}{\sqrt{2}}G_{(5)}(\alpha)\left(\frac{1}{a}||\gamma||^2+a||\nabla_{(5)}\beta||^2\right). \quad (53)$$

An alternative bound may be obtained from (50) and the inequalities

$$|D_{0,y}\beta_{i,j}|\leq\frac{1}{2\Delta s}(|\beta_{i,j+1}|+|\beta_{i,j-1}|),\ |D_{0,x}\beta_{i,j}|\leq\frac{1}{2\Delta s}(|\beta_{i+1,j}|$$

$$+|\beta_{i,j-1}|),\text{ etc.:}$$

$$[\gamma,\mathcal{J}^1_{(9)}(\alpha,\beta)]\leq G_{(5)}(\alpha)\sum_{i=1}^{L-1}\sum_{j=1}^{M-1}\Delta s^2|\gamma_{i,j}|\frac{1}{6}\{|D_{0,y}\beta_{i+1,j}|$$

$$+|D_{0,y}\beta_{i-1,j}|$$

$$+|D_{0,x}\beta_{i,j+1}|+|D_{0,x}\beta_{i,j-1}|+\frac{1}{\Delta s}(|\beta_{i,j+1}|+|\beta_{i,j-1}|+|\beta_{i+1,j}|$$

$$+|\beta_{i-1,j}|)$$

$$+2|D_{0,y}\beta_{i,j}|+2|D_{0,x}\beta_{i,j}|\}\leq G_{(5)}(\alpha)\sum_{i=1}^{L-1}\sum_{j=1}^{M-1}\frac{\Delta s}{6}|\gamma_{i,j}|\{|\beta_{i+1,j+1}|$$

$$+|\beta_{i+1,j-1}|+|\beta_{i-1,j+1}|+|\beta_{i-1,j-1}|+2|\beta_{i,j+1}|+|\beta_{i,j-1}|$$

$$+|\beta_{i+1,j}|+|\beta_{i-1,j}|)\}\leq G_{(5)}(\alpha)\frac{1}{\Delta s}\left(\frac{1}{a}||\gamma||^2+a||\beta||^2\right) \quad (54)$$

if $\beta=0$ at the boundary.

If the Laplacian $\nabla^2_{(9)}$ is used instead of $\nabla^2_{(5)}$, it is easy to show from

$$\mathcal{J}^1_{(9)}(\alpha,\beta)=\frac{1}{12}\{D_{+\xi}\alpha_{i,j}D_{+\eta}\beta_{i+1,j}+D_{+\xi}\alpha_{i-1,j-1}D_{+\eta}\beta_{i,j-1}$$

$$+D_{+\xi}\alpha_{i,j-1}D_{+\eta}\beta_{i+1,j-1}+D_{+\xi}\alpha_{i-1,j}D_{+\eta}\beta_{i,j}-D_{+\eta}\alpha_{i+1,j}D_{+\xi}\beta_{i,j}$$

$$-D_{+\eta}\alpha_{i,j-1}D_{+\xi}\beta_{i-1,j-1}-D_{+\eta}\alpha_{i+1,j-1}D_{+\xi}\beta_{i,j-1}$$

$$-D_{+\eta}\alpha_{i,j}D_{+\xi}\beta_{i-1,j}+2(D_{0,x}(\alpha D_{0,y}\beta)_{i,j}-D_{0,y}(\alpha D_{0,x}\beta)_{i,j}$$

$$+D_{0,y}(\beta D_{0,x}\alpha)_{i,j}-D_{0,x}(\beta D_{0,y}\alpha)_{i,j})+4(D_{0,x}\alpha_{i,j}D_{0,y}\beta_{i,j}$$

$$-D_{0,y}\alpha_{i,j}D_{0,x}\beta_{i,j})\} \quad (55)$$

that (53) has its exact counterpart with

$$G_{(9)}(\alpha)\equiv\max_{i,j}(\max(|D_{+x}\alpha|,|D_{+y}\alpha|,|D_{+\xi}\alpha|,|D_{+\eta}\alpha|)) \quad (56)$$

and the validity of (54) with this $G(\alpha)$ follows immediately.

Note added in proof: a closer study indicates that the extrapolation procedure $(\nabla^2\Psi^k)_B=(\nabla^2\Psi^{k+1}+\nabla^2\Psi^{k-1})_{B+1}-\nabla^2\Psi^k{}_{B+2}$, where the indices B, $B+1$, $B+2$ denote the boundary point and the two first interior points normal to the boundary, be a better alternative at outflow points.

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